Lecture 3: September 4

Let us first finish up the discussion from last time. Recall that X was a compact Kähler manifold of dimension n, with Kähler form ω. We defined an action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the space of smooth forms on X (and hence on the cohomology of X) by setting

$$X = 2\pi i\,\omega\colon A^k(X,\mathbb{C}) \to A^{k+2}(X,\mathbb{C})$$
$$Y = (2\pi i)^{-1}\Lambda\colon A^k(X,\mathbb{C}) \to A^{k+2}(X,\mathbb{C}),$$

and by letting H act on $A^k(X,\mathbb{C})$ as multiplication by $k-n$. The point was the resulting action is independent of the choice of $i=\sqrt{-1}$. We then described the induced action of the element

$$w = e^X e^{-Y} e^X \in \mathrm{SL}_2(\mathbb{C})$$

in terms of the Lefschetz decomposition, and we proved the following key formula, which relates w and the Hodge $*$-operator: for any $\alpha \in A^{p,q}(X)$, one has

$$*\alpha = (2\pi)^k \cdot \frac{(-1)^q \varepsilon(k)}{(2\pi i)^n} \cdot w\alpha,$$

where $k = p+q$ and $\varepsilon(k) = (-1)^{k(k-1)/2}$. We can now use this to construct a polarization for the Hodge structure on each $H^k(X,\mathbb{C})$. Namely, suppose that $\alpha,\beta \in A^{p,q}(X)$. Then $\overline{\beta} \in A^{q,p}(X)$, and therefore

$$*\overline{\beta} = \frac{1}{(2\pi i)^n} \cdot (-1)^p \varepsilon(k)(2\pi)^k \cdot w\overline{\beta},$$

where again $k = p+q$. If we put this into the formula for the hermitian inner product on $A^k(X,\mathbb{C})$ induced by the hermitian metric h, we get

$$h(\alpha,\beta) = \int_X \alpha \wedge *\overline{\beta} = (-1)^p \varepsilon(k)(2\pi)^k \cdot \frac{1}{(2\pi i)^n} \int_X \alpha \wedge w\overline{\beta}.$$

The conclusion is that

$$(-1)^p \varepsilon(k) \cdot \frac{1}{(2\pi i)^n} \int_X \alpha \wedge w\overline{\beta} = (2\pi)^{-k} h(\alpha,\beta)$$

is positive definite on $A^{p,q}(X)$. According to our definition, this means exactly that the hermitian pairing

$$(\alpha,\beta) \mapsto \varepsilon(k) \frac{1}{(2\pi i)^n} \int_X \alpha \wedge w\overline{\beta},$$

polarizes the Hodge structure on $H^k(X,\mathbb{C})$.

Hodge-Lefschetz structures. In the cohomology algebra

$$H^*(X,\mathbb{C}) = \bigoplus_{k=0}^{2n} H^k(X,\mathbb{C}),$$

each summand has a Hodge structure of weight k, and the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts on the whole thing. This is an example of a so-called "Hodge-Lefschetz structure". Since the same kind of structure also appears in the study of degenerating variations of Hodge structure, we are going to dwell a bit on the definition. We will also see that this gives a nice way to understand the formula for the polarization that we have just derived.

Let us set $V_k = H^{n+k}(X,\mathbb{C})$; this has a Hodge structure of weight $n+k$, and its weight with respect to the action by H is equal to k. Note that the weight of the Hodge structures and the weight with respect to the $\mathfrak{sl}_2(\mathbb{C})$-action are off by $n = \dim X$. Now consider the graded vector space

$$V = \bigoplus_{k\in\mathbb{Z}} V_k,$$

with the induced action by the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the Lie group $\mathrm{SL}_2(\mathbb{C})$. As we saw in Lecture 1, the operator

$$X\colon V_k \to V_{k+2}(1)$$

is a morphism of Hodge structures of weight k, due to the fact that ω is a closed $(1,1)$-form; likewise,

$$Y\colon V_k \to V_{k-2}(-1)$$

is a morphism of Hodge structures of weight k. In other words, the action of $\mathfrak{sl}_2(\mathbb{C})$ is compatible with the Hodge structures on the weight spaces V_k.

Definition 3.1. Let $n \in \mathbb{Z}$. A *Hodge-Lefschetz structure of central weight* n is a finite-dimensional graded complex vector space

$$V = \bigoplus_{k\in\mathbb{Z}} V_k$$

with an action by the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, subject to the following conditions:

(a) Each V_k is the k-eigenspace of H, hence $X(V_k) \subseteq V_{k+2}$ and $Y(V_k) \subseteq V_{k-2}$.

(b) Each V_k has a Hodge structure of weight $n+k$, and

$$X\colon V_k \to V_{k+2}(1) \quad \text{and} \quad Y\colon V_k \to V_{k-2}(-1)$$

are morphisms of Hodge structure.

For the time being, suppose that V is an abstract Hodge-Lefschetz structure of central weight n. When we reviewed the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ in Lecture 2, we saw that $X^k\colon V_{-k} \to V_k$ is an isomorphism for every $k \geq 1$.

Lemma 3.2. *For every $k \geq 1$, the morphism*

$$X^k\colon V_{-k} \to V_k(k)$$

is an isomorphism of Hodge structures of weight $n-k$.

Proof. This is a general fact about Hodge structures: If $f\colon H_1 \to H_2$ is a morphism of Hodge structures of some weight, and if f is an isomorphism of vector spaces, then f^{-1} is also a morphism of Hodge structures. The proof is left as an exercise. □

Consider now the induced action of $w = e^X e^{-Y} e^X \in \mathrm{SL}_2(\mathbb{C})$ on V. We know that $w\colon V_k \to V_{-k}$ is an isomorphism for every $k \in \mathbb{Z}$. But since X and Y are only morphisms up to a Tate twist, each term in the series expansion of $w = e^X e^{-Y} e^X$ needs a different Tate twist, and so it is not immediately clear that w is a morphism of Hodge structures. We should check that this is actually the case.

Proposition 3.3. *If V is a Hodge-Lefschetz structure of central weight n, then*

$$w\colon V_k \to V_{-k}(-k)$$

is an isomorphism of Hodge structures (for every $k \in \mathbb{Z}$).

Proof. For the same reason as above, it is enough to show that

$$w\colon V_k \to V_{-k}(-k)$$

is a morphism of Hodge structures of weight $n+k$. This can be done with the help of Lefschetz decompositions. Any $a \in V_k$ has a unique Lefschetz decomposition

$$a = \sum_{j\geq \max(k,0)} \frac{X^j}{j!} a_j,$$

where $a_j \in V_{k-2j}$ satisfies $Y a_j = 0$. Recall that only terms with $j \geq k$ appear in the sum, because $X^{2j-k+1} a_j = 0$, which implies that $X^j a_j = 0$ for $j < k$.

Let us first show that if $a \in V_k^{p,q}$, where $p + q = n + k$, then also $a_j \in V_{k-2j}^{p-j,q-j}$. Let ℓ be the maximal integer with the property that $a_\ell \neq 0$ but $a_{\ell+1} = 0$; this exists because V is finite-dimensional. Since $X^{2j-k+1} a_j = 0$ for every j, we get

$$\frac{X^{2\ell-k}}{\ell!} a_\ell = X^{\ell-k} a \in V_{2\ell-k}^{p+\ell-k,q+\ell-k},$$

due to the fact that X is a morphism. But by the preceding lemma,

$$X^{2\ell-k} \colon V_{k-2\ell} \to V_{2\ell-k}(2\ell - k)$$

is an isomorphism of Hodge structures, and therefore $a_\ell \in V_{k-2\ell}^{p-\ell,q-\ell}$. We can now subtract off the term involving a_ℓ, and continue by descending induction on j, to prove that $a_j \in V_{k-2j}^{p-j,q-j}$ for every $j \geq \max(k, 0)$.

It is now easy to show that $wa \in V_{-k}^{p-k,q-k}$. Indeed, Proposition 2.5 gives

$$wa = \sum_{j \geq \max(k,0)} w \frac{X^j}{j!} a_j = \sum_{j \geq \max(k,0)} (-1)^j \frac{X^{j-k}}{(j-k)!} a_j \in V_{-k}^{p-k,q-k}.$$

This says exactly that w is a morphism of Hodge structures. □

Polarized Hodge-Lefschetz structures. Now let us revisit the question of polarizations, starting again from the geometric case where $V_k = H^{n+k}(X, \mathbb{C})$. For each $k \in \mathbb{Z}$, we have a bilinear pairing

$$S_k \colon V_k \otimes_{\mathbb{C}} \overline{V_{-k}} \to \mathbb{C}, \quad S_k(\alpha, \beta) = \varepsilon(n-k) \cdot \frac{1}{(2\pi i)^n} \int_X \alpha \wedge \overline{\beta}. \tag{3.4}$$

The extra sign factor $\varepsilon(n-k) = (-1)^{(n-k)(n-k-1)/2}$ is there to make the formulas nicer; this will become clear in a moment.

Note. For any complex vector space W, we denote by $\overline{W}$ the conjugate complex vector space; it has the same underlying set, but the scalar multiplication by $\lambda \in \mathbb{C}$ is defined as $\lambda \cdot w = \bar{\lambda} w$. Therefore a bilinear pairing

$$S \colon W \otimes_{\mathbb{C}} \overline{W} \to \mathbb{C}$$

is the same thing as a pairing on W that is linear in the first argument and conjugate-linear in the second argument.

These pairings interact very well with the $\mathfrak{sl}_2(\mathbb{C})$-action.

Proposition 3.5. *The pairings in (3.4) have the following properties:*

(a) $S_k(H\alpha, \beta) = -S_k(\alpha, H\beta)$ *for all* $\alpha \in V_k$ *and* $\beta \in V_{-k}$.
(b) $\overline{S_k(\alpha, \beta)} = S_{-k}(\beta, \alpha)$ *for all* $\alpha \in V_k$ *and* $\beta \in V_{-k}$.
(c) $S_{k+2}(X\alpha, \beta) = S_k(\alpha, X\beta)$ *for all* $\alpha \in V_k$ *and* $\beta \in V_{-k-2}$.
(d) $S_{-k}(w\alpha, \beta) = S_k(\alpha, w\beta)$ *for all* $\alpha \in V_k$ *and* $\beta \in V_k$.
(e) $S_{k-2}(Y\alpha, \beta) = S_k(\alpha, Y\beta)$ *for all* $\alpha \in V_k$ *and* $\beta \in V_{-k+2}$.

Proof. Since $H\alpha = k\alpha$ and $H\beta = -k\beta$, the identity in (a) is trivial. The proof of (b) is also straightforward: α is an $(n+k)$-form, β an $(n-k)$-form, and so

$$\begin{aligned} \overline{S_k(\alpha, \beta)} &= \varepsilon(n-k) \frac{(-1)^n}{(2\pi i)^n} \int_X \overline{\alpha} \wedge \beta = \varepsilon(n-k) \frac{(-1)^n}{(2\pi i)^n} \int_X (-1)^{(n+k)(n-k)} \beta \wedge \overline{\alpha} \\ &= (-1)^k \varepsilon(n-k) \frac{1}{(2\pi i)^n} \int_X \beta \wedge \overline{\alpha} = \varepsilon(n+k) \frac{1}{(2\pi i)^n} \int_X \beta \wedge \overline{\alpha} = S_{-k}(\beta, \alpha), \end{aligned}$$

due to the fact that $\varepsilon(n+k) = (-1)^k \varepsilon(n-k)$.

To prove (c), we use the fact that $X = 2\pi i\, L$. This gives

$$\begin{aligned} S_{k+2}(X\alpha, \beta) &= \varepsilon(n-k-2)\frac{1}{(2\pi i)^n}\int_X 2\pi i\,\omega \wedge \alpha \wedge \overline{\beta} \\ &= \varepsilon(n-k)\frac{1}{(2\pi i)^n}\int_X \alpha \wedge \overline{2\pi i\,\omega \wedge \beta} = S_k(\alpha, X\beta), \end{aligned}$$

due to the fact that $\varepsilon(n-k) = -\varepsilon(n-k-2)$.

Now let us prove (d). It is enough to consider the case where $\alpha, \beta \in V_k^{p,q}$ for some $p+q = n+k$. The formula for the Hodge $*$-operator in Proposition 2.7 reads

$$w\alpha = \frac{(2\pi i)^n(-1)^q\varepsilon(n+k)}{(2\pi)^{n+k}} *\alpha,$$

and therefore

$$S_{-k}(w\alpha, \beta) = \frac{(-1)^q}{(2\pi)^{n+k}}\int_X *\alpha \wedge \overline{\beta} = \frac{(-1)^{n+k+q}}{(2\pi)^{n+k}}\int_X \overline{\beta} \wedge *\alpha = \frac{(-1)^p}{(2\pi)^{n+k}}\overline{h(\beta, \alpha)}$$

is, up to a constant factor, equal to the hermitian inner product on forms induced by the Kähler metric h. But the right-hand side equals

$$\frac{(-1)^p}{(2\pi)^{n+k}}h(\alpha, \beta) = \overline{S_{-k}(w\beta, \alpha)} = S_k(\alpha, w\beta),$$

using the identity we have just derived, together with (b).

The least obvious statement is (e), because it is not clear at first glance how one can move $Y = (2\pi i)^{-1}\Lambda$ from one factor of the integral to the other. Here the fact that $wXw^{-1} = -Y$ comes to the rescue. Recall first that $w^2\alpha = (-1)^k\alpha$, which means that $w^{-1}\alpha = (-1)^k w\alpha$. Using the identities we have already derived, we get

$$S_{k-2}(Y\alpha, \beta) = (-1)^{k+1}S_{k-2}(wXw\alpha, \beta) = (-1)^{k+1}S_k(\alpha, wXw\beta) = S_k(\alpha, Y\beta),$$

because similarly $w^{-1}\beta = (-1)^{-k+2}w\beta$. This completes the proof. □

These identities become much easier to remember if we combine all the individual pairings in (3.4) into one big sesquilinear pairing

$$S\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}, \quad S\big|_{V_k \otimes_{\mathbb{C}} \overline{V_\ell}} = \begin{cases} S_k & \text{if } \ell = -k, \\ 0 & \text{otherwise.} \end{cases}$$

This removes the needs for any subscripts. With this notation in place, the identities in Proposition 3.5 are saying that the pairing S is hermitian symmetric, and that

$$\begin{aligned} S \circ (H \otimes \mathrm{id}) &= -S \circ (\mathrm{id} \otimes H), \\ S \circ (X \otimes \mathrm{id}) &= S \circ (\mathrm{id} \otimes X), \\ S \circ (Y \otimes \mathrm{id}) &= S \circ (\mathrm{id} \otimes Y), \\ S \circ (w \otimes \mathrm{id}) &= S \circ (\mathrm{id} \otimes w). \end{aligned} \tag{3.6}$$

In other words, the three operators X, Y, and w are self-adjoint with respect to the hermitian pairing S, whereas H is anti self-adjoint.

We can now redo the calculation from the beginning of class. Suppose that $\alpha, \beta \in A^{p,q}(X)$ represent two cohomology classes in $V_k = H^{n+k}(X, \mathbb{C})$, so that $p+q = n+k$. During the proof of Proposition 3.5, we showed that

$$S_k(\alpha, w\beta) = \frac{(-1)^p}{(2\pi)^{n+k}}h(\alpha, \beta),$$

where h is the positive definite hermitian inner product induced by the Kähler metric. This is saying exactly that the hermitian pairing

$$S_k \circ (\mathrm{id} \otimes w)\colon V_k \otimes_{\mathbb{C}} \overline{V_k} \to \mathbb{C}$$

is a polarization of the Hodge structure on $V_k = H^{n+k}(X, \mathbb{C})$. We can say this even more concisely as follows: the graded vector space V is a direct sum of Hodge structures of different weights, and the single pairing $S \circ (\mathrm{id} \otimes w)$ polarizes all of these Hodge structures at the same time.

As before, this suggest the following general definition.

Definition 3.7. Let V be a Hodge-Lefschetz structure of central weight n. Then a *polarization* of V is a hermitian pairing

$$S \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$$

with the following two properties:

(a) The four identities in (3.6) are satisfied.
(b) The hermitian pairing $S \circ (\mathrm{id} \otimes w)$ polarizes the Hodge structure of weight $n + k$ on each V_k.

Suppose that S is a polarization of V, in the sense we have just defined. The relation $S \circ (H \otimes \mathrm{id}) = -S \circ (\mathrm{id} \otimes H)$ implies that

$$S\big|_{V_k \otimes_{\mathbb{C}} \overline{V_\ell}} = 0 \quad \text{unless } \ell = -k,$$

and so S is actually given by a collection of sesquilinear pairings $S_k \colon V_k \otimes_{\mathbb{C}} \overline{V_{-k}} \to \mathbb{C}$. The second condition in the definition is then saying that the hermitian pairing

$$S_k \circ (\mathrm{id} \otimes w) \colon V_k \otimes_{\mathbb{C}} \overline{V_k} \to \mathbb{C}$$

polarizes the Hodge structure on V_k. The following sequence of exercises shows that most of these conditions, with the exception of positivity, can be expressed in functorial language.

Exercise 3.1. If H has a Hodge structure of weight n, then the conjugate complex vector space $\overline{H}$ also has a Hodge structure of weight n, with

$$\overline{H}^{p,q} = \overline{H^{q,p}}.$$

Now suppose that V is a Hodge-Lefschetz structure of central weight n. Show that the conjugate complex vector space

$$\overline{V} = \bigoplus_{k \in \mathbb{Z}} \overline{V_k}$$

is also Hodge-Lefschetz structure of central weight n, where the action of H is unchanged, but where X and Y act with an extra minus sign. (This is dictated by the geometric case, where $X = 2\pi i\, L$ and $Y = (2\pi i)^{-1}\Lambda$.)

Exercise 3.2. Suppose that V' and V'' are Hodge-Lefschetz structures of central weight n' and n''. Show that the tensor product $V' \otimes_{\mathbb{C}} V''$ is naturally a Hodge-Lefschetz structure of central weight $n' + n''$: to be precise,

$$\left(V' \otimes_{\mathbb{C}} V''\right)_k = \bigoplus_{i+j=k} V'_i \otimes_{\mathbb{C}} V''_j,$$

and the $\mathfrak{sl}_2(\mathbb{C})$-action is given by the usual formulas

$$\begin{aligned}
X(v' \otimes v'') &= Xv' \otimes v'' + v' \otimes Xv'', \\
Y(v' \otimes v'') &= Yv' \otimes v'' + v' \otimes Yv'', \\
H(v' \otimes v'') &= Hv' \otimes v'' + v' \otimes Hv'', \\
w(v' \otimes v'') &= wv' \otimes wv''.
\end{aligned}$$

Exercise 3.3. We can turn $\mathbb{C}(-n)$ into a Hodge-Lefschetz structure of central weight $2n$ by letting $\mathfrak{sl}_2(\mathbb{C})$ act trivially. Let V be a Hodge-Lefschetz structure of central weight n, and $S\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ a hermitian pairing. Show that

$$S\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}(-n)$$

is a morphism of Hodge-Lefschetz structures of central weight $2n$ if, and only if, S satisfies the identities in (3.6) and the Hodge decomposition on each V_k is orthogonal with respect to the pairing $S_k \circ (\mathrm{id} \otimes w)$.